Some demonstrations on the functions of the second degree

Maxime LUCE

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Abstract

We will prove the formula that allows to solve a polynomial equation of degree 2 as well as the canonical form of a function of the second degree and some rules of sum and products on the roots.

1 Canonical form

Theorem 1.1

Any second-degree trinomial of developed form $P(x) = ax^2 + bx + c, (b,c) \in \mathbb{R}^2$ and $a \in \mathbb{R} \backslash \{0\}$, $\forall x \in \mathbb{R}$ is uniquely written in the form :

$$P(x) = a(x-\alpha)^2 + \beta \text{ with } \alpha = \frac{-b}{2a} \text{ and } \beta = P(\alpha) = -\frac{b^2-4ac}{4a}$$

Proof 1.1

$(b,c) \in \mathbb{R}^2$ and $a \in \mathbb{R}\backslash\{0\}$, $\forall x \in \mathbb{R}$:

$$\begin{aligned}
P(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - a \times \left(\frac{b}{2a}\right)^2 + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{ab^2}{4a^2} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a} \\
&= a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \left(-\frac{b^2-4ac}{4a}\right) \\
&= a\left(x-\alpha\right)^2 + \beta \text{ with } \alpha = -\frac{b}{2a} \text{ and } \beta = P(\alpha) = -\frac{b^2-4ac}{4a}
\end{aligned}$$

▲

2 Solving second-degree equations

Theorem 2.1

Let P be a trinomial of the second degree defined on $\mathbb{R}$ by $P(x) = ax^2 + bx + c$, $(b, c) \in \mathbb{R}^2$ and $a \in \mathbb{R}\backslash\{0\}$. The discriminant of the polynomial P is called the real $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, polynomial P has two distinct roots $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$.
- If $\Delta = 0$, polynomial P has single root $x = \frac{-b}{2a}$
- If $\Delta < 0$, polynomial P has no real roots.

But the trinomial $az^2 + bz + c$ with $(b, c) \in \mathbb{R}^2$ and $a \in \mathbb{R}\backslash\{0\}$ has two complex roots combined $z_1 = \frac{-b-i\sqrt{\Delta}}{2a}$ and $z_2 = \frac{-b+i\sqrt{\Delta}}{2a}$ when $\Delta < 0$.

Proof 2.1

We start from the canonical form shown above. We're going to solve $P(x) = 0$, where $P(x) = ax^2 + bx + c$, $(b, c) \in \mathbb{R}^2$ and $a \in \mathbb{R}\backslash\{0\}$. Let $x \in \mathbb{R}$:

$$\begin{aligned}
& P(x) = 0 \\
\Leftrightarrow\ & ax^2 + bx + c = 0 \\
\Leftrightarrow\ & a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} = 0 \qquad \text{(by the canonical form)} \\
\Leftrightarrow\ & a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a} \\
\Leftrightarrow\ & \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{(as } a \neq 0\text{)}
\end{aligned}$$

Here, there are two possibilities: either $b^2 - 4ac$ is positive or zero, or it is negative.

If $b^2 - 4ac > 0$:

$$\begin{aligned}\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ \Leftrightarrow x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\ \Leftrightarrow x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \\ \Leftrightarrow x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\ \Leftrightarrow x &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a} \\ \Leftrightarrow x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\end{aligned}$$

If we let $\Delta = b^2 - 4ac$, we get :

$$\Leftrightarrow x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

If $b^2 - 4ac < 0$:

$$\begin{aligned}\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ \Leftrightarrow \left(x + \frac{b}{2a}\right)^2 &= i^2 \frac{-(b^2 - 4ac)}{4a^2} \qquad (\text{as } i^2 = -1\ ) \\ \Leftrightarrow x + \frac{b}{2a} &= \pm i\sqrt{\frac{-\Delta}{4a^2}} \qquad (\text{as } \frac{-\Delta}{4a^2} \geq 0\ ) \\ \Leftrightarrow x &= \frac{\pm i\sqrt{-\Delta}}{2a} - \frac{b}{2a} \\ \Leftrightarrow x &= \frac{-b \pm i\sqrt{-\Delta}}{2a}\end{aligned}$$

If we resume, if $\Delta > 0$, we get two distincts roots $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$.
If $\Delta = 0$, we get $x_1 = \frac{-b-\sqrt{0}}{2a}$ and $x_2 = \frac{-b+\sqrt{0}}{2a} \Leftrightarrow x_1 = x_2 = \frac{-b}{2a}$.
If $\Delta < 0$, we get two complex roots : $z_1 = \frac{-b-i\sqrt{\Delta}}{2a}$ and $z_2 = \frac{-b+i\sqrt{\Delta}}{2a}$ when $\Delta < 0$ ▲

2.1 General formula

Lemma 2.1

If $Z = a + ib \in \mathbb{C}^*$, then the equation $z^2 = Z$ admits two opposite solutions in $\mathbb{C}$.

Proof 2.2

Let's find out if there is $z = x + iy$ such as $z^2 = Z$. We've got the equivalencies:

$$\begin{aligned}\left((x+iy)^2 = a+ib\right) &\Longleftrightarrow \begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases} \\ &\Longleftrightarrow \begin{cases} x^2 - y^2 = a \\ x^2 + y^2 = \sqrt{a^2+b^2} \\ 2xy = b \end{cases} \\ &\Longleftrightarrow \begin{cases} x = \pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} \\ y = \pm sign(b)\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}, \end{cases}\end{aligned}$$

with $sign(b) = \begin{cases} 1 & \text{if } b > 0 \\ 0 & \text{if } b = 0 \\ -1 & \text{if } b < 0 \end{cases}$

The result is then deduced ▲

Theorem 2.2

Let $(a, b, c) \in \mathbb{C}^3$ (with $a \neq 0$) and $\Delta = b^2 - 4ac \in \mathbb{C}$. Then the equation $az^2 + bz + c = 0$, noted (E') in the following, admits two solutions in $\mathbb{C}$, given by :

- If $\Delta = 0$, $z_1 = z_2 = -\dfrac{b}{2a}$.
- If $\Delta \neq 0$, then
$$z_1 = \frac{-b+\delta}{2a} \qquad \text{and} \qquad z_2 = \frac{-b-\delta}{2a},$$
where δ is such that $\delta^2 = \Delta$.

Proof 2.3

$$\begin{aligned}(E') \Leftrightarrow a\left[\left(z+\frac{b}{2a}\right)^2 - \frac{\Delta}{2a}\right] = 0 \\ \Leftrightarrow a\left[\left(z+\frac{b}{2a}\right)^2 - \left(\frac{\delta}{2a}\right)^2\right] = 0 \\ \Leftrightarrow a\left(z - \frac{-b+\delta}{2a}\right)\left(z - \frac{-b-\delta}{2a}\right) = 0.\end{aligned}$$

If $\Delta = 0$, then $\delta = 0$ and $z_1 = z_2 = \frac{-b}{2a}$. Otherwise, the lemma above ensures that δ such that $\delta^2 = \Delta$ exist, and from then on, we've got:

$$z_1 = \frac{-b+\delta}{2a} \qquad \text{and} \qquad z_2 = \frac{-b-\delta}{2a}$$

▲

Remark. This result generalizes the well-known formulas when a, b and c are real. Indeed:

If $\Delta > 0$, then $\Delta = (\sqrt{\Delta})^2$ and we can take $\delta = \sqrt{\Delta}$. We therefore get:

$$x_1 = \frac{-b-\sqrt{\Delta}}{2a} \qquad \text{and} \qquad x_2 = \frac{-b+\sqrt{\Delta}}{2a}$$

If $\Delta = 0$, then $\Delta = 0^2$ and we can take $\delta = 0$. We therefore get:

$$x_1 = x_2 = \frac{-b}{2a}$$

If $\Delta < 0$, then $\Delta = -(-\Delta) = i^2(-\Delta) = (i^2\sqrt{\Delta})$ (because $-\Delta > 0$) and we can take $\delta = i\sqrt{-\Delta}$. We therefore get:

$$z_1 = \frac{-b+i\sqrt{-\Delta}}{2a} \qquad \text{and} \qquad z_2 = \frac{-b-i\sqrt{-\Delta}}{2a}$$

3 Sum and root products

Theorem 3.1

Let P be a polynomial function defined by $P(x) = ax^2 + bx + c$, $(b,c) \in \mathbb{R}^2$ and $a \in \mathbb{R}\backslash\{0\}$ such as $\Delta \geq 0$. Equation $P(x) = 0$ admits two distinct or combined roots x_1 and x_2 that verify that : $\begin{cases} x_1 + x_2 & = \frac{-b}{a} \\ x_1 \times x_2 & = \frac{c}{a} \end{cases}$

Proof 3.1

Let $x_1 = \frac{-b+\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b-\sqrt{\Delta}}{2a}$ and $\Delta = b^2 - 4ac$.

$$\begin{aligned} x_1 + x_2 &= \frac{-b+\sqrt{\Delta}}{2a} + \frac{-b-\sqrt{\Delta}}{2a} \\ &= \frac{-2b}{2a} \\ &= \frac{-b}{a} \end{aligned}$$

$$\begin{aligned} x_1 \times x_2 &= \frac{-b+\sqrt{\Delta}}{2a} \times \frac{-b-\sqrt{\Delta}}{2a} \\ &= \frac{(-b+\sqrt{\Delta})(-b-\sqrt{\Delta})}{4a^2} \\ &= \frac{b^2 - \Delta}{4a^2} \\ &= \frac{b^2 - b^2 + 4ac}{4a^2} \\ &= \frac{c}{a} \end{aligned}$$

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References

[1] N. Nguyen, E. Schneider and S. Daniel, *Prépas Sciences, Spécialité Maths*, (2019).

[2] N. Nguyen, E. Schneider and S. Daniel, *Prépas Sciences, Maths*, (2017).

[3] Online CAPEC Course