Some demonstrations on the sequels

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Abstract

We will prove the explicit forms of the geometrical and arithmetical sequences and prove the formulas of the sums n first squares and cubes.

1 Geometric and arithmetic sequences

1.1 Arithmetic

Definition 1.1

$\mathbb{R}^{\mathbb{N}}$ designates the function space : $\mathbb{N} \longrightarrow \mathbb{R}$. They will be called real (values) sequences.

Definition 1.2

Let $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. $(u_n)_{n\in\mathbb{N}}$ is an arithmetic sequence if and only if

$$\exists r \in \mathbb{R}, \forall n \in \mathbb{N}, u_{n+1} = u_n + r$$

Definition 1.3

$\mathbb{R}_a^{\mathbb{N}}$ refers to the set of arithmetic sequences.

$$\mathbb{R}_a^{\mathbb{N}} = \{(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} | \exists r \in \mathbb{R}, \forall n \in \mathbb{N}, u_{n+1} = u_n + r\}$$

Theorem 1.1: Explicit form of arithmetic sequences

$$(u_n)_{n\in\mathbb{N}} \in \mathbb{R}_a^{\mathbb{N}} \Leftrightarrow \exists r \in \mathbb{R}, \forall n \in \mathbb{N}, u_n = u_0 + nr$$

Proof 1.1

Proposition 1.1.1. The sequence (u_n) is an arithmetic sequence of first term $u0$ and reason r.

Proof. Let $(u_n)_{n\in\mathbb{N}} | u_n = u_0 + nr$

We have : $u_{n+1} - u_n = u_0 + (n+1)r - (u_0 + nr) = u_0 + nr + r - u_0 - nr = r$

The sequence (u_n) is thus an arithmetic sequence of first term u_0 and reason r. ∎

Proposition 1.1.2. The sequence (u_n) can be described as $u_n = u_0 + nr$.

Proof. Let (u_n) an arithmetic sequence of reason r and first term u_0.

$$\begin{align}
u_1 &= u_0 + r \tag{1}\\
u_2 &= u_1 + r \tag{2}\\
... &= ... \tag{3}\\
u_{n-1} &= u_{n-2} + r \tag{4}\\
u_n &= u_{n-1} + r \tag{5}
\end{align}$$

By adding member to member all these equalities, we obtain :

$$\begin{align}
u_1 + u_2 + ... + u_{n-1} + u_n &= u_0 + u_1 + ... + u_{n-1} + nr \tag{6}\\
u_n &= u_0 + u_1 - u_1 + u_2 - u_2 + ... + u_{n-1} - u_{n-1} + nr \tag{7}\\
u_n &= u_0 + nr \tag{8}
\end{align}$$

∎

Proof Theorem. We can see that Proposition 1.1.1. $\Rightarrow$ Proposition 1.1.2. and Proposition 1.1.2. $\Rightarrow$ Proposition 1.1.1.. It means that Proposition 1.1.1. $\Leftrightarrow$ Proposition 1.1.2..

This proves that only the writing $u_n = u_0 + nr$ is an arithmetic sequence of reason r and first term u_0. ∎

∎

1.2 Geo

Definition 2.4

$\mathbb{R}^{\mathbb{N}}$ designates the function space : $\mathbb{N} \longrightarrow \mathbb{R}$. They will be called real (values) sequences.

Definition 2.5

Let $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. $(u_n)_{n\in\mathbb{N}}$ is an geometric sequence if and only if

$$\exists q \in \mathbb{R}, \forall n \in \mathbb{N}, u_{n+1} = u_n \times q$$

Definition 2.6

$\mathbb{R}_g^{\mathbb{N}}$ refers to the set of geometric sequences.

$$\mathbb{R}_g^{\mathbb{N}} = \{(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} | \exists q \in \mathbb{R}, \forall n \in \mathbb{N}, u_{n+1} = u_n \times q\}$$

Theorem 2.2: Explicit form of geometric sequences

$$(u_n)_{n\in\mathbb{N}} \in \mathbb{R}_g^{\mathbb{N}} \Leftrightarrow \exists q \in \mathbb{R}, \forall n \in \mathbb{N}, u_n = u_0 \times q^n$$

Proof 2.2

Proposition 1.2.1. The sequence (u_n) is an geometric sequence of first term $u0$ and reason q.

Proof. Let $(u_n)_{n\in\mathbb{N}} | u_n = u_0 \times q^n$

We have : $\dfrac{u_{n+1}}{u_n} = \dfrac{u_0 \times q^{n+1}}{u_0 \times q^n} = \dfrac{u_0 \times q^n \times q}{u_0 \times q^n} = q$ ∎

Proposition 1.2.2. The sequence (u_n) can be described as $u_n = u_0 \times q^n$.

Proof. Let (u_n) a geometric sequence of reason q and first term u_0.

$$u_1 = u_0 q \tag{9}$$
$$u_2 = u_1 q \tag{10}$$
$$... = ... \tag{11}$$
$$u_{n-1} = u_{n-2} q \tag{12}$$
$$u_n = u_{n-1} q \tag{13}$$

By multiplying member to member all these equalities, we obtain :

$$u_1 \times u_2 \times ... \times u_{n-1} \times u_n = u_0 \times u_1 \times ... \times u_{n-1} \times q^n \tag{14}$$

$$u_n = u_0 \times \frac{u_1 \times ... \times u_{n-1}}{u_1 \times ... \times u_{n-1}} \times q^n \tag{15}$$

$$u_n = u_0 q^n \tag{16}$$

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Proof Theorem. We can see that Proposition 1.2.1. ⇒ Proposition 1.2.2. and Proposition 1.2.2. ⇒ Proposition 1.2.1.. It means that Proposition 1.2.1. ⇔ Proposition 1.2.2..

This proves that only the writing $u_n = u_0 \times q^n$ is a geometric sequence of reason q and first term u_0. ■

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2 Some sums

2.1 Early sums

Theorem 1.1

$\forall n \in \mathbb{N}^*$:

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Proof 1.1

$\forall n \in \mathbb{N}^*$:

$$\begin{aligned}
\text{Let : } S = \quad & 1 + 2 + ... + n-2 + n-1 + n && (17)\\
\text{We have also : } S = \quad & n + n-1 + ... + 3 + 2 + 1 && (18)
\end{aligned}$$

By adding member to member the two ties, we obtain :

$$2S = \underbrace{(n+1) + (n+1) + (n+1) + ... + (n+1) + (n+1) + (n+1)}_{n \text{ times}} \quad (19)$$

$$2S = n(n+1) \quad (20)$$

$$S = \frac{n(n+1)}{2} \quad (21)$$

$\blacksquare$

Remark. This demonstration was discovered by Gauss when, as a child, his mistress asked him to sum the numbers from 1 to 100.

Theorem 1.2

Let q be any real number and n a natural number :

- If $q \neq 1$ then

$$1 + q + q^2 + ... + q^n = \frac{1 - q^{n+1}}{1 - q}$$

  or

$$\sum_{k=0}^{n} q^k = \frac{1 - q^{n+1}}{1 - q}$$

- If $q = 1$ then

$$1 + q + q^2 + ... + q^n = n + 1$$

  or

$$\sum_{k=0}^{n} q^k = n + 1$$

Proof 1.2

If $q \neq 1$ then let $S = 1 + q + q^2 + ... + q^n$. So we have

$$q \times S = q \times (1 + q + q^2 + ... + q^n) \tag{22}$$
$$= q + q^2 + q^3 + ... + q^{n+1} \tag{23}$$

Hence :

$$S - q \times S = (1 + q + q^2 + ... + q^n) - (q + q^2 + q^3 + ... + q^{n+1}) \tag{24}$$
$$S - q \times S = 1 + q - q + q^2 - q^2 + ... + q^n - q^n + q^{n+1} \tag{25}$$
$$S - q \times S = 1 + q^{n+1} \tag{26}$$
$$S \times (1 - q) = 1 + q^{n+1} \tag{27}$$
$$S = \frac{1 - q^{n+1}}{1 - q} \tag{28}$$

And for $q = 1$,

$$S = 1 + \underbrace{q + q^2 + ... + q^n}_{n} = \underbrace{1 + 1 + 1 + ... + 1}_{n+1} = n + 1$$

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2.2 Square sums

Theorem 2.3: Sum of the first n squares

$\forall n \in \mathbb{N}$:

$$1^2 + 2^2 + 3^2 + ... + n^2 = \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof 2.3

Let : $S_1 = 1 + 2 + 3 + ... + n$, $S_2 = 1^2 + 2^2 + 3^2 + ... + n^2$ and $S_3 = 1^3 + 2^3 + 3^3 + ... + n^3$. $\forall n \in \mathbb{N}$:

$$(n+1)^3 = (n+1)(n+1)^2 = (n+1)(n^2 + 2n + 1) = n^3 + 3n^2 + 3n + 1$$

$$(0+1)^3 = 0^3 + 3 \times 0^2 + 3 \times 0 + 1 \tag{29}$$
$$(1+1)^3 = 1^3 + 3 \times 1^2 + 3 \times 1 + 1 \tag{30}$$
$$(2+1)^3 = 2^3 + 3 \times 2^2 + 3 \times 2 + 1 \tag{31}$$
$$... = ... \tag{32}$$
$$(n-2+1)^3 = (n-2)^3 + 3 \times (n-2)^2 + 3 \times (n-2) + 1 \tag{33}$$
$$(n-1+1)^3 = (n-1)^3 + 3 \times (n-1)^2 + 3 \times (n-1) + 1 \tag{34}$$
$$(n+1)^3 = n^3 + 3 \times n^2 + 3 \times n + 1 \tag{35}$$
$$\tag{36}$$

By adding member to member the two ties, we obtain :

$$
\begin{aligned}
S_3 + (n+1)^3 &= S_3 + 3S_2 + 3S_1 + (n+1) && (37)\\
(n+1)^3 &= 3S_2 + 3S_1 + (n+1) && (38)\\
n^3 + 3n^2 + 3n + 1 &= 3S_2 + 3\frac{n(n+1)}{2} + (n+1) && (39)\\
2n^3 + 6n^2 + 6n + 2 &= 6S_2 + 3n(n+1) + 2(n+1) && (40)\\
2n^3 + 6n^2 + 6n + 2 - 3n^2 - 3n - 2n - 2 &= 6S_2 && (41)\\
6S_2 &= 2n^3 + 3n^2 + n && (42)\\
6S_2 &= n(2n^2 + 3n + 1) && (43)\\
6S_2 &= n(n+1)(2n+1) && (44)\\
S_2 &= \frac{n(n+1)(2n+1)}{6} && (45)
\end{aligned}
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Proof 2.4: By Mathematical induction

Let's show by recurrence on $n \in \mathbb{N}$ that

$$P_n : \text{"} \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \text{"}$$

<u>Initialization:</u> $n = 0$

$$\sum_{k=0}^{0} k^2 = \frac{0 \times 1 \times 1}{6} = \frac{0}{6} = 0$$

Let $n \in \mathbb{N}$ such as P_n

$$
\begin{aligned}
\sum_{k=0}^{n+1} k^2 &= \sum_{k=0}^{n} k^2 + (n+1)^2 && (46)\\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad \text{(by the recurrence hypothesis)} && (47)\\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} && (48)\\
&= \frac{n(2n^2 + n + 2n + 1) + 6n^2 + 12n + 6}{6} && (49)\\
&= \frac{2n^3 + n^2 + 2n^2 + n + 6n^2 + 12n + 6}{6} && (50)\\
&= \frac{2n^3 + 9n^2 + 13n + 6}{6} && (51)\\
&= \frac{(n+1)(n+2)(2n+3)}{6} && (52)
\end{aligned}
$$

Hence P_{n+1}
So we have $\forall n \in \mathbb{N}$:

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

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2.3 Cube

Theorem 3.4: Sum of the first n cubes

$\forall n \in \mathbb{N}$:

$$1^3 + 2^3 + 3^3 + ... + n^3 = \sum_{k=0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Proof 3.5

et $S_1 = 1+2+3+...+n$, $S_2 = 1^2+2^2+3^2+...+n^2$, $S_3 = 1^3+2^3+3^3+...+n^3$ and $S_4 = 1^4+2^4+3^4+...+n^4$.
$\forall n \in \mathbb{N}$:

$$(n+1)^4 = (n+1)^2(n+1)^2 = (n^2+2n+1)(n^2+2n+1) = n^4+4n^3+6n^2+4n+1$$

$$\begin{aligned}
(0+1)^4 &= 0^4 + 4\times 0^3 + 6\times 0^2 + 4\times 0 + 1 && (53)\\
(1+1)^4 &= 1^4 + 4\times 1^3 + 6\times 1^2 + 4\times 1 + 1 && (54)\\
(2+1)^4 &= 2^4 + 4\times 2^3 + 6\times 2^2 + 4\times 2 + 1 && (55)\\
... &= ... && (56)\\
(n-2+1)^4 &= (n-2)^4 + 4(n-2)^3 + 6(n-2)^2 + 4(n-2) + 1 && (57)\\
(n-1+1)^4 &= (n-1)^4 + 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1 && (58)\\
(n+1)^4 &= n^4 + 4n^3 + 6n^2 + 4n + 1 && (59)\\
& && (60)
\end{aligned}$$

By adding member to member the two ties, we obtain :

$$\begin{aligned}
S_4 + (n+1)^4 &= S_4 + 4S_3 + 6S_2 + 4S_1 + (n+1) && (61)\\
(n+1)^4 &= 4S_3 + 6S_2 + 4S_1 + (n+1) && (62)\\
(n+1)^4 &= 4S_3 + n(n+1)(2n+1) + 4\frac{n(n+1)}{2} + (n+1) && (63)\\
(n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - (n+1) &= 4S_3 && (64)\\
(n+1)((n+1)^3 - n(2n+1) - 2n - 1) &= 4S_3 && (65)\\
4S_3 &= (n+1)(n^3+n^2) && (66)\\
4S_3 &= (n+1)n^2(n+1)^2 && (67)\\
S_3 &= \frac{n^2(n+1)^2}{4} && (68)\\
S_3 &= \left(\frac{n(n+1)^2}{2}\right)^2 = S_1^2 && (69)
\end{aligned}$$

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Proof 3.6: By Mathematical induction

Let's show by recurrence on $n \in \mathbb{N}$ that

$$P_n : " \sum_{k=0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 "$$

<u>Initialization:</u> $n = 0$

$$\sum_{k=0}^{0} k^3 = \left(\frac{0 \times 1}{2}\right)^2 = 0$$

Let $n \in \mathbb{N}$ such as P_n

$$\sum_{k=0}^{n+1} k^3 = \sum_{k=0}^{n} k^3 + (n+1)^3 \tag{70}$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \qquad \text{(by the recurrence hypothesis)} \tag{71}$$

$$= \frac{n^2(n^2+2n+1)}{4} + (n+1)^3 \tag{72}$$

$$= \frac{n^4 + 2n^3 + n^2 + 4(n^3 + 2n^2 + n + n^2 + 2n + 1)}{4} \tag{73}$$

$$= \frac{n^4 + 2n^3 + n^2 + 4n^3 + 8n^2 + 4n + 4n^2 + 8n + 4}{4} \tag{74}$$

$$= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} \tag{75}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2 \tag{76}$$

Hence P_{n+1}
So we have $\forall n \in \mathbb{N}$:

$$\sum_{k=0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

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